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Dynamics of Relativistic Solitons Due to Pseudo Sine-Gordon Equation

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Abstract This paper is a review of some of the most important concepts and phenomena for relativistic solitons. The Sine-Gordon equation and the pseudo Sine-Gordon equations are studied in this context. A final discussion is conducted in the for soliton in more than one spatial dimensions.

Keywords Relativistic solitons · Sine-Gordon equation · Solitons interaction · Non-linear physics · Solitons theory

1 Introduction

Some physicists are still hopeful that one type of a non-linear field theory finally lead to their old wish about fields and particles unification and can express the properties of each one [1]. As the result, some important attempts were made which probably the most important of them is chiral solitons innovation [2–8].

In the Skyrme model, baryons are solitons of a mesonic non-linear field, and the baryonic number is a quantized quantity, which is obtained from a topologic current [2–8]. Although, the Skyrme model can anticipate baryons characters only with accuracy of 20%, but in recent four decades it kept its importance as an approximate pattern of quantum chromodynamics, for many colors [2–8].

In relativistic non-linear fields, solitons are stable and substitute answers of field equations with particle characters. Fortunately, some of these characters are being discussed.

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Though, it seems that many characters of these solutions are still undiscovered because of difficulty of applying non-linear equations. However, applying numerical methods and computational calculations probably are good solutions [9–14].

2 The Sine-Gordon Equation

The Lagrangian density:

$$L = \frac{1}{2}\partial^{\mu}\phi\partial_{\mu}\phi - \frac{a}{b}[1 - \cos(b\phi)] \tag{1}$$

where ϕ is a real scalar field which leads to below dynamical equation in $1 + 1$ dimensions (a position dimension and a time dimension) (assume that $c = 1$):

$$\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial t^2} = a\sin(b\phi) \tag{2}$$

This equation is called as the Sine-Gordon equation. In the limit of small oscillations, the equation approaches to the Klein-Gordon equation for a real field. It will be simply shown that a stationary solution of mentioned equation is in this form:

$$\phi_{(x)} = \frac{4}{b}\tan^{-1} e^{\sqrt{ab}(x-x_0)} \tag{3}$$

Single-soliton (kink) solutions of the Sine-Gordon equation can be obtained using direct method. These solutions can be converted to moving solitons using the Lorentz transformation:

$$\phi_{(x,t)} = \frac{4}{b}\tan^{-1} e^{\gamma\sqrt{ab}(x-x_0-vt)} + 2n\pi \tag{4}$$

where, v is soliton velocity, $\gamma = (1 - v^2)^{-1/2}$, x_0 is initial position of soliton, and n is an arbitrary integer number.

Single-soliton and multiple-soliton solutions can also be obtained using the Bäcklund transforms. First, introducing new variables:

$$\zeta = \frac{1}{2}\sqrt{ab}(x - t), \tag{5}$$

$$\tau = \frac{1}{2}\sqrt{ab}(x + t), \tag{6}$$

$$\sigma = b\phi \tag{7}$$

We can reform the Sine-Gordon equation, with respect to light-cone co-ordinates as:

$$\sigma_{\varsigma\tau} = \sin\sigma \tag{8}$$

It can be simply shown that if we transform the Sine-Gordon equation by the Bäcklund transformation it will be invariant:

$$\sigma'_{\zeta} = \sigma_{\zeta} - 2\beta\sin\left(\frac{\sigma + \sigma'}{2}\right), \tag{9}$$

$$\sigma'_{\tau} = -\sigma_{\tau} + \frac{2}{\beta}\sin\left(\frac{\sigma - \sigma'}{2}\right), \tag{10}$$

$$\zeta' = \zeta, \tag{11}$$

$$\tau' = \tau \tag{12}$$

In other words, the Sine-Gordon equation rules on σ and σ'. In these transformations, β which is called the Bäcklund parameter is a constant and arbitrary quantity which is related to solitons velocity. Applying these transformations it will be possible to obtain multi-solitons solutions from solutions of fewer solitons, sequentially. The substitution theorem only makes possible to extract soliton solution just with algebraic method.

3 The Substitution Theorem

If σ_{n-1} is a solution of the Sine-Gordon equation, and σ_n' and σ_n'' are two other solutions obtained from σ_{n-1} with the Bäcklund parameters of β' and β'', a higher order solution can be obtained in this form:

$$\sigma_{n+1} = 4\tan^{-1}\left\{\frac{\beta' + \beta''}{\beta' - \beta''}\tan\left(\frac{\sigma_n' - \sigma_n''}{4}\right)\right\} + \sigma_{n-1} \tag{13}$$

4 Solitons Interaction

The two solitons, when coming close to each other, interact with each other because of non-linear effects. After collision, solitons separate from each other and go away while keep their initial form. If the mentioned model is integrable, existence of indefinite conservative quantities will cause to revive initial form of solitons and avoid energy propagating into the medium. The Sine-Gordon equation is one of these models. In contrast, model ϕ^4 with the Lagrangian density as:

$$L = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \lambda(\phi^2 - \phi_0^2)^2 \tag{14}$$

Despite of owning stationary solitary solution, does not have this important property. The force between solitons can be calculated both by numerical method and analytically by approximation in long distances. It can be shown that in approximation in long distances, solitons of the Sine-Gordon equation apply a decreasing exponential force to each other, which is equivalent to the Born approximation in the quantum field theory, and is created by means of exchanging of massive intermediate particles.

There is a question: how do solitons of the Sine-Gordon equation behave when they face an external force field? Numerical and analytical methods can be used to answer this question. If in the Sine-Gordon equation, one of these parameters a and b or both of them are so slow functions of position, single-soliton solutions will face an effective potential which is obtained from the below equation (a_0 and b_0 are values of parameters in a reference position):

$$U_{(x)} = 8\frac{a_0^{1/2}}{b_0^{3/2}}\left(\frac{a(x)^{1/2}b_0^{3/2}}{b(x)^{3/2}a_0^{1/2}} - 1\right) \tag{15}$$

For obtaining this effective potential the below equation is used:

$$E = E_0 + T + U \tag{16}$$

where in the above equation, E is total energy of soliton, E_0 is rest energy, T is kinetic energy, and U is effective potential. Facing potential U, soliton behaves so similar to a classic particle. For instance, it keeps moving to a place where T is zero (turning point). Then it returns and retrieves its initial velocity in opposite direction.

Numerical calculations, confirm this case, and show that the mentioned approximation will be true even in the case of rapid changes of potential. Despite these similarities, there are two important differences with behavior of a classic particle:

1. When a high energy soliton faces the external potential, lose some of their energy in forms of propagating low amplitude oscillations, or non-linear agitations (like soliton anti-soliton pairs or breathers).
2. When a soliton faces a thin potential dam is able to pass through it, even if T is lower than U_0 height of potential dam. This phenomenon has been reported for solitons of the non-linear Schrödinger equation (NLSE) [15]. The difference between this classic tunneling phenomenon and its quantum equivalent is that, in spite of quantum status, the probability of passing is zero or one and no other probability ($0 < P < 1$) is observed.

5 The Generalizations from the Sine-Gordon Equation

If we enter a power coefficient in the Lagrangian of the Sine-Gordon equation:

$$L = \phi^n \left\{ \frac{(n+2)^2}{8} \partial^\mu \phi \partial_\mu \phi - (1 - \cos\phi) \right\} \tag{17}$$

Then we will find the pseudo-Sine-Gordon equation:

$$\Box\psi = \frac{\partial U}{\partial \psi}, \tag{18}$$

$$\psi = \phi^{\frac{n}{2}+1}, \tag{19}$$

$$U(\psi) = 2\psi^{\frac{2n}{n+2}} \sin^2\left(\frac{1}{2}\psi^{\frac{2}{n+2}}\right) \tag{20}$$

This equation has solitary solutions in form:

$$\phi_{(x,t)} = \pm 4 \tan^{-1} e^{\frac{2\gamma(x-vt)}{n+2}} + 4m\pi; \quad m \in Z \tag{21}$$

which their rest energy can be obtained using the below equation:

$$E_m = \int_{\psi_m}^{\psi_{m\neq 1}} \sqrt{2U(\psi)}d\psi; \quad (v = 0) \tag{22}$$

The limited value of rest energy (mass) of these solutions when m approaches an extremely value, is described as:

$$E_m \approx 8(4m\pi)^{\frac{n}{n+2}} \tag{23}$$

The mentioned solutions own below characters which make them different from solitons of the Sine-Gordon equation:

1. Solutions related to different sectors (different ms) have different masses. Therefore, mass divergence, which occurs for solitons of the Sine-Gordon equations, does not exist in this case.
2. Different boundary conditions ($\phi(x \to \pm\infty)$) avoid a solitary wave of sector m_1 passing a solitary wave of sector m_2.
3. Generally, solutions construct solitary waves which despite of dynamical stability, will be annihilated when collide with their anti-waves. This result has been obtained by solving the dynamical equation, numerically.

Another generalization form of the Sine-Gordon equation is changing the potential of self interaction, in this way:

$$\circ^{a} \frac{\circ}{-a} \varepsilon^{\mu\alpha\beta} \varepsilon_{abc} \phi_a \partial_\alpha \phi_b \partial_\beta \phi_c, \tag{24}$$

$$\text{If: } \phi \geq 0, \quad a = a_1 \tag{25}$$

and

$$\text{If: } \phi < 0, \quad a = a_2 \tag{26}$$

Dynamical equation and momentum-energy tensor in this potential is principally equal to the Sine-Gordon equation, except that the related parameter of different amplitudes of field changes is different.

By choosing the potential in the method mentioned above, causes to produce two solitons with different masses. A soliton anti-soliton pair with high level of mass if collide each other, convert to a pair with lower level of mass (Fig. 1). While collision of a low mass pair leads to their propagation, unless the energy of their mass center is high enough; therefore, a bounded massive pair is created for a short time (depends on initial energy of pair) and then they will be annihilated again and a pair will be created similar to first pair (Fig. 2). If the Lagrangian parameters are chosen properly, solitons with very low rest mass are accessible. This causes that by annihilation of a heavy pair, a pair almost without mass is created which leaves the site of interaction with velocity near to the velocity of light (Fig. 1).

t

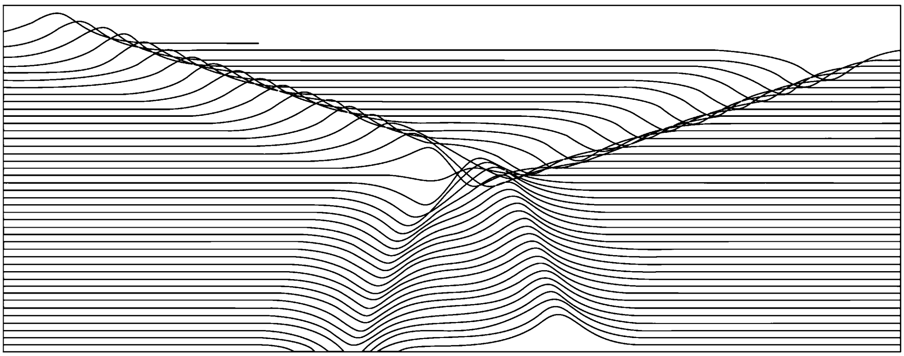

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Fig. 1 Collision of a soliton anti-soliton pair with high rest mass causes creation of a pair with lower mass which leaves the interaction site with higher velocity (the pseudo-Sine-Gordon equation, second type). In this numerical test, let: $v_1 = -v_2 = 0.12$, $a_1 = 1$, and $a_2 = 0.0002$

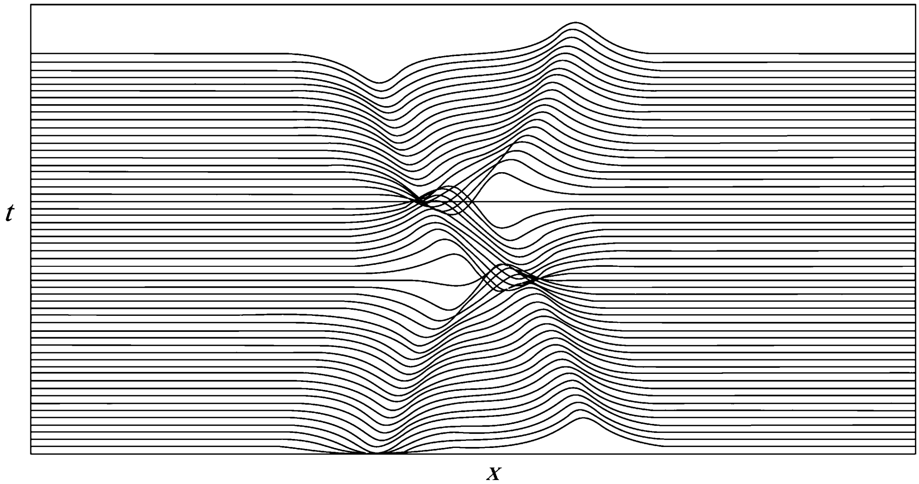

Fig. 2 If the energy of a soliton anti-soliton pair related to the pseudo-Sine-Gordon equation (second type) is high enough, it will temporarily yield a bound massive pair which annihilates after a short time and creates a pair similar to first pair. In this numerical test, we let $v_1 = -v_2 = 0.12$, $a_1 = 1$, and $a_2 = 1.1$

6 Results and Discussion

6.1 The Non-linear $O(3)$ Model

The Lagrangian density is as below:

$$L = \frac{1}{2}\partial^{\mu}\phi_a\partial_{\mu}\phi_a; \quad a = 1, 2, 3 \tag{27}$$

where ϕ_a is a three-components scalar field, leads to a linear dynamical equation with $O(3)$ symmetry. Applying below constraint:

$$\phi_a\phi_a = 1 \tag{28}$$

Between the components of field while maintaining the mentioned symmetry character, we convert the model to a non-linear model [16]. This constraint will enter the action by applying a Lagrangian coefficient. Consequently, the field equation will be in this form (sum on repeated subscripts):

$$\Box\phi_a - (\phi_b\Box\phi_b)\phi_{a=0} \tag{29}$$

This field has interesting specifications in two spatial dimensions. As usual, total energy of system is obtained using time component integration of momentum-energy tensor. Therefore, for stationary solutions of below equation:

$$E = \frac{1}{2}\int \partial_i\phi_a\partial_i\phi_a d^2x \tag{30}$$

Classical loses of this system are obtained using the condition $\partial_i\phi_a = 0$ (or $\phi_a = \phi_a^{(0)}$ and therefore $E = 0$). This condition occurs at a point on sphere $\phi_a\phi_a = 1$ in internal space (instinctive break down of symmetry).

The substituted solutions of this model which own limited energy, comply with this condition:

$$\text{If: } r \to \infty, r|\nabla \phi_a| \to 0 \tag{31}$$

or

$$\lim_{r\to\infty} \phi_a = \phi_a^{(0)} \tag{32}$$

Since field vector is constant for distances far from the origin, as far as it relates to the dynamics of mentioned field, all of these points, in indefinite, can be assumed as one point.

In this case, positional space R^2 can be mapped to a two-dimensional sphere (S^2), by a stereographic mapping. Consequently, a homotopic group π_2 will be hold between sphere S^2 of field lose and positional space sphere:

$$\pi_2(S^2) = Z \tag{33}$$

This group is isomorphic with integer numbers set under summation operation [17]. By describing current density in this form:

$$J^\mu = \frac{1}{8\pi}\varepsilon^{\mu\alpha\beta}\varepsilon_{abc}\phi_a\partial_\alpha\phi_b\partial_\beta\phi_c \tag{34}$$

It can be easily shown that, this current is conservative:

$$\partial_\mu J^\mu = \frac{1}{8p}\varepsilon^{\mu\alpha\beta}\varepsilon_{abc}\partial_\mu\phi_a\partial_\alpha\phi_b\partial_\beta\phi_c\alpha \det\left|\frac{\partial\phi_a}{\partial x^\mu}\right| = 0 \tag{35}$$

The final result can be obtained from the fact that the three components of ϕ_a are not independent. We can also conclude that, total charge related to this current is quantized:

$$\begin{aligned} Q &= \int J^0 d^2x \\ &= \frac{1}{8\pi}\int \varepsilon_{\mu v}\varepsilon_{abc}\phi_a \frac{\partial\phi_b}{\partial x^\mu}\frac{\partial\phi_c}{\partial x^v} d^2x \\ &= \frac{1}{8\pi}\int \varepsilon_{rs}\varepsilon_{abc}\frac{\partial\phi_b}{\partial\zeta_r}\frac{\partial\phi_c}{\partial\zeta_s} d^2\zeta \\ &= \frac{1}{4\pi}\int ds^{(int)} = n \end{aligned} \tag{36}$$

In these equations, variables ζ_1 *and* ζ_2 are angular coordinates which show one point of surface of sphere S^2 in internal space. It can be shown that stationary solutions with an arbitrary positive charge can be extracted using below equation [16]:

$$\omega(z) = \left(\frac{z - z_0}{\lambda}\right)^n \tag{37}$$

where:

$$\omega = \omega_1 + \omega_2, \tag{38}$$

$$\omega_1 = \frac{2\phi_1}{1 - \phi_3}, \tag{39}$$

$$\omega_2 = \frac{2\phi_2}{1-\phi_3}, \tag{40}$$

$$z = x + iy \tag{41}$$

Also, λ is an arbitrary real number and n is a positive integer number which total energy and total charge are proportioned to it.

6.2 The Skyrme Model

The Skyrme [2, 18] showed that it is possible to begin by a chiral non-linear bosonic field, and reach fermions that are the solitons of the model, indeed. This model which is concluded from the below Lagrangian, is generalized form of the non-linear sigma model:

$$L = \frac{F_\pi^2}{16} Tr(\partial_\mu U \partial^\mu U^+) + \frac{1}{32e^2} Tr\{[U^+\partial_\mu U, U^+\partial_v U][U^+\partial^\mu U, U^+\partial^v U]\} \tag{42}$$

In this Lagrangian, U is a member of group $SU(2)$:

$$U = e^{\frac{2i}{F_\pi}\tau_a\pi_a} = \phi_0 + i\tau_a\phi_a, \tag{43}$$

$$\phi_0 = \cos\frac{2\pi}{F_\pi}, \tag{44}$$

$$\phi_a = \hat{\pi}_a \sin\left(\frac{2\pi}{F_\pi}\right) \tag{45}$$

and ϕ_0 and ϕ_a are components of a scalar field which sweeps a sphere S^3 in the space of the field. In these equations, $\pi = (\pi_a\pi_a)^{1/2}$ and $\widehat{\pi}_a = \pi_a/\pi$, and pion propagation constant is $F_p = 186$ MeV. Here, as in extreme far distances, the field approaches a constant value; all points can be mapped to one point, in indefinite. Therefore, by means of a mapping process like stereographic mapping, positional three-dimensional space can be reduced to a sphere S^3. Now, each solution from field equations can be considered as a mapping process from positional S^3 to internal S^3 (fields space).

A set of these mappings construct a homotopic group π_3 which is homomorphic with group of integer numbers under summation operation. Here, topologic current:

$$B^\mu = \frac{1}{24p^2}\varepsilon^{\mu v\alpha\beta} Tr[(U^+\partial_v U)(U^+\partial_\alpha U)(U^+\partial_\beta U)] \tag{46}$$

is regional conservative:

$$\partial_\mu B^\mu = 0 \tag{47}$$

and its related charge which is quantized indicates baryon number of system, based on the Skyrme's hypothesis:

$$B = \int B^0 d^3x \tag{48}$$

By quantizing this model by means of summative coordinates [19], some of baryons physical characters (such as mass, spin, isospin, and magnetic bipolar momentum) can be extracted by an accuracy about 20%.

6.3 The Electrodynamics Isovector Model

In this model, we start by using a Lagrangian which owns below specifications:

1. Relativistic invariance
2. Instinctive break-down of symmetry $O(3)$
3. Stability of solitary solutions
4. Energy conditions obeying by the Hamiltonian density. For instance, consider below Lagrangian:

$$L = -\lambda(\partial^{\mu}\phi_a\partial_{\mu}\phi_a)^2 - b_0\left(1 - 2\frac{\phi}{\phi_0}\right)^2 \tag{49}$$

It can be studied as a testing model. In this Lagrangian, ϕ_a is an isovector field (components are in pseudo-scalar form) and λ, b_0, and ϕ_0 are constant, real, and positive coefficients.

Solitary solutions of this model can be obtained by choosing $\phi_a = \phi_{(r)}\frac{x^a}{r}$ and substituting in dynamic equations.

Since complexity of non-linear equations in three dimensions avoid us obtaining the solutions analytically, it is possible to obtain the desired solutions by variant method. Boundary solutions are in this form:

$$\text{If: } r \to \infty, \quad \phi_{(r)} \approx \phi_0\left(1 - \frac{8\phi_0^4}{b_0 r^4} + \cdots\right) \tag{50}$$

and

$$\text{If: } r \to 0, \quad \phi_{(r)} \approx k\frac{r}{r_0} - \frac{1}{10k^2}\left(\frac{r}{r_0}\right)^2 + \cdots \tag{51}$$

where k is a constant coefficient which its value is obtained from compatibility of boundary solutions.

Total energy which was calculated numerically is in this form:

$$E = 4.05 \times 4^{3/4}\pi b_0^{1/4}\phi_0^3$$

Dynamical stability of solitary solutions can be proved by applying low-amplitude perturbations, and studying energy changes.

Now, by using below definition for anti-symmetric tensor $F^{\mu v}$:

$$F^{\mu v} = \frac{4\pi}{c}\varepsilon^{\mu v\alpha\beta}\varepsilon_{abc}\phi_a\partial_{\alpha}\phi_b\partial_{\beta}\phi_c \tag{52}$$

It can be shown that we may reach equations like the Maxwell equations:

$$\partial_{\mu}F^{\mu v} = \frac{4\pi}{c}J^v \tag{53}$$

$$J^v = \varepsilon^{\mu v\alpha\beta}\varepsilon_{abc}\partial_{\mu}\phi_a\partial_{\alpha}\phi_b\partial_{\beta}\phi_c \tag{54}$$

In this equation, J^v is current four-vector which is conservative:

$$\partial_{\gamma}J^v = 0 \tag{55}$$

Also, by defining twin tensor $F^{\mu v}$ in ordinary form:

$$\tilde{F}^{\mu v} = \frac{1}{2}\varepsilon^{\mu v\alpha\beta} F_{\alpha\beta} \tag{56}$$

It can be easily shown that, this tensor obeys below equation:

$$\partial_\mu \tilde{F}^{\mu v} = \frac{4\pi}{c}\tilde{J}^v \tag{57}$$

In this equation, $\widehat{J^v}$ is magnetic current density, which is also conservative. In order to eliminate magnetic current (because of not observing magnetic monopolars), it is possible to use an auxiliary vector field. Based on this model, we can justify the quantized characteristic of electrical charge by means of using homotopic group π_2. Finally, we should mention that, the presented model, leads to a radius about the classic radius of charged particles. Also, quantizing solitary solutions by summative coordinates is not possible because of inertial momentum tending to indefinite.

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